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Note

Numerical Solution of a Nonlinear Klein–Gordon Equation

We compute the solutions of the equation $u_{tt} - \Delta u + m^2 u + gu^p = 0$ for p odd and $m, g > 0$. Our computations show that (i) the solutions remain bounded as $t \to \infty$, (ii) the amplitude decreases as p increases, and (iii) the number of oscillations increases as p increases. Because of (i), theoretical results imply that the amplitude goes to zero like $O(t^{-3/2})$ as $t \to \infty$.

1. Analysis

The nonlinear Klein–Gordon equation (NLKG)

$$u_{tt} - \Delta u + m^2 u + G'(u) = 0, \tag{1}$$

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \Psi(x) \tag{2}$$

($x \in \mathbb{R}^3$, $m > 0$) is probably the simplest nonlinear relativistic equation of mathematical physics. A complete understanding of it would illuminate our view of many other such equations. The most important property of a solution is that the energy is constant:

$$E = \int \left(\frac{1}{2} u_t^2 + \frac{1}{2} \mid \nabla u \mid^2 + \frac{m^2}{2} u^2 + G(u)\right) dx. \tag{3}$$

Assuming $G \geqslant 0$, each term in this expression is bounded by E for all time. However, it does not automatically follow that the amplitude

$$M(t) = \max_x \mid u(x, t) \mid$$

must be bounded.

The mathematical theory of (1) consists of three major parts. Assume arbitrary initial data $\phi(x)$ and $\Psi(x)$ which are smooth and small at infinity (so that E is finite). Assume $G'(u) = \mid u \mid^{p-1} u$. (i) If $p < 5$, a unique smooth solution exists for all time (see [2]); its amplitude is bounded. (ii) If $p \geqslant 5$, a weak solution exists for all time (see [5]), but it is not known whether it is unique. If a weak solution has bounded amplitude, it is smooth and unique. (iii) For any $p > 8/3$ and for solutions of bounded

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amplitude, there is a scattering theory; in particular, they decay uniformly as fast as free solutions:

$$M(t) \leqslant c(1 + |t|)^{-3/2} \tag{4}$$

(see [3]). The major gap in the theory is the boundedness of the solutions when $p \geqslant 5$.

Not only is NLKG a model for other relativistic physical systems, it is also a prototype of a large class of nonrelativistic systems which find themselves in the same predicament as in (ii) above. The most famous example is the nonstationary Navier–Stokes equations in three space dimensions, where global existence was proved by J. Leray over four decades ago but uniqueness of the weak solutions is still an open problem.

We have computed some solutions of NLKG for various values of p. The computed solutions are indeed bounded.

D. Jesperson and J. Rauch have done similar computations which are in general agreement with our results (personal communication).

For the computation we consider radial solutions of (1), $u = u(r, t)$, $r = |x|$ and smooth initial data ϕ, Ψ of compact support. We take $m = 1$ and $G'(u) = |u|^{p-1} u$. Thus

$$u_{tt} - u_{rr} - \frac{2}{r} u_r + u + |u|^{p-1} u = 0. \tag{5}$$

Putting $v = ru$,

$$v_{tt} - v_{rr} + v + r^{1-p} |v|^{p-1} v = 0, \qquad 0 \leqslant r < \infty,$$
$$v(0, t) = 0.$$

The energy is

$$E = 4\pi \int_0^\infty \left\{ \frac{1}{2} (u_t^2 + u_r^2 + u^2) + \frac{1}{p+1} |u|^{p+1} \right\} r^2 \, dr = 4\pi E',$$

$$E' = \int_0^\infty \left\{ \frac{1}{2} (v_t^2 + v_r^2 + v^2) + \frac{|v|^{p+1}}{(p+1) r^{p-1}} \right\} dr \tag{7}$$

because

$$v_r^2 = (ru_r + u)^2 = r^2 u_r^2 + (ru^2)_r .$$

From (7), we have

$$(v^2)_r = 2v \, v_r \leqslant v_r^2 + v^2,$$

$$v^2 \leqslant \int_r^\infty (v_r^2 + v^2) \, dr \leqslant 2E'.$$

Hence $|u(r, t)| \leqslant (2E')^{1/2}/r$. Thus, if there were an unbounded radial solution, it would have to become unbounded near the origin $r = 0$.

2. Computation

We used almost the simplest of all finite-difference schemes: central second differences for the terms v_{tt} and v_{rr}. The scheme is

$$\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{(\Delta t)^2} - \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta r)^2} + \frac{1}{2}[v_j^{n+1} + v_j^{n-1}] + \frac{1}{(j\,\Delta r)^{p-1}} \frac{G(v_j^{n+1}) - G(v_j^{n-1})}{v_j^{n+1} - v_j^{n-1}} = 0. \tag{8}$$

Here $G'(v) = |v|^{p-1} v$, $G(v) = |v|^{p+1}/(p+1)$.

When we tried the simpler explicit scheme with the nonlinear term $G'(v_j^n)$, we found some computed solutions rapidly blowing up, even in the case $G'(v) = v^3$. This indicates a numerical instability because Jörgens' theorem implies the boundedness of the solutions, as mentioned above.

On the other hand, for scheme (8) the computed solutions are bounded: see below. In fact, scheme (8) has the advantage that there is a discrete energy which is constant. We thank K. W. Morton for his help with the spatial derivative term. The discrete energy is

$$\frac{E_n}{\Delta r} = \frac{1}{2}\sum_j \left(\frac{v_j^{n+1} - v_j^n}{\Delta t}\right)^2 + \frac{1}{2}\sum_j \left(\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta r}\right)\left(\frac{v_{j+1}^n - v_j^n}{\Delta r}\right) + \frac{1}{2}\sum_j \frac{(v_j^{n+1})^2 + (v_j^n)^2}{2} + \sum_j \frac{G(v_j^{n+1}) + G(v_j^n)}{2(j\,\Delta r)^{p-1}}. \tag{9}$$

If (8) is multiplied by $\frac{1}{2}(v_j^{n+1} - v_j^{n-1})$, the identity $E_n = E_{n-1}$ results. Thus $E_n = E_0$ for $n = 0, 1, 2, \ldots$, and the scheme appears to be stable.

If $G = 0$, the equation is linear, the scheme is explicit, and it is stable if

$$\left(\frac{\Delta t}{\Delta r}\right)^2 < 1 + \frac{1}{4}(\Delta t)^2. \quad \text{(See [4].)}$$

At each time-step, the scheme (8) requires solving a simple functional equation for the unknown v_j^{n+1}. We used Newton's method to accomplish this. We chose the mesh sizes $\Delta t = \Delta r = 0.002$.

For $G'(v) = |v|^{p-1} v$, $p = 2, 3, 4, \ldots$, the computed solutions appear finite and stable. For $G'(v) = v^2, v^4$, etc., the computed solutions blow up in a finite time. This behavior corroborates the mathematical theory (see [1]).

In Fig. 1 is presented a typical time evolution for the nonlinear term $G'(u) = u^7$ at successive time intervals, $t = 0.00, 0.04, 0.08, 0.12, 0.16, 0.20$. The solution $u = r^{-1}v$ is plotted versus $r = |x|$. The initial data for this run are $\phi = h(r)$, $\Psi = h'(r) + h(r)/r$, where

$$h(r) = 5 \exp 100[1 - (1 - (10r - 1)^2)^{-1}], \; f(r) = rh(r).$$

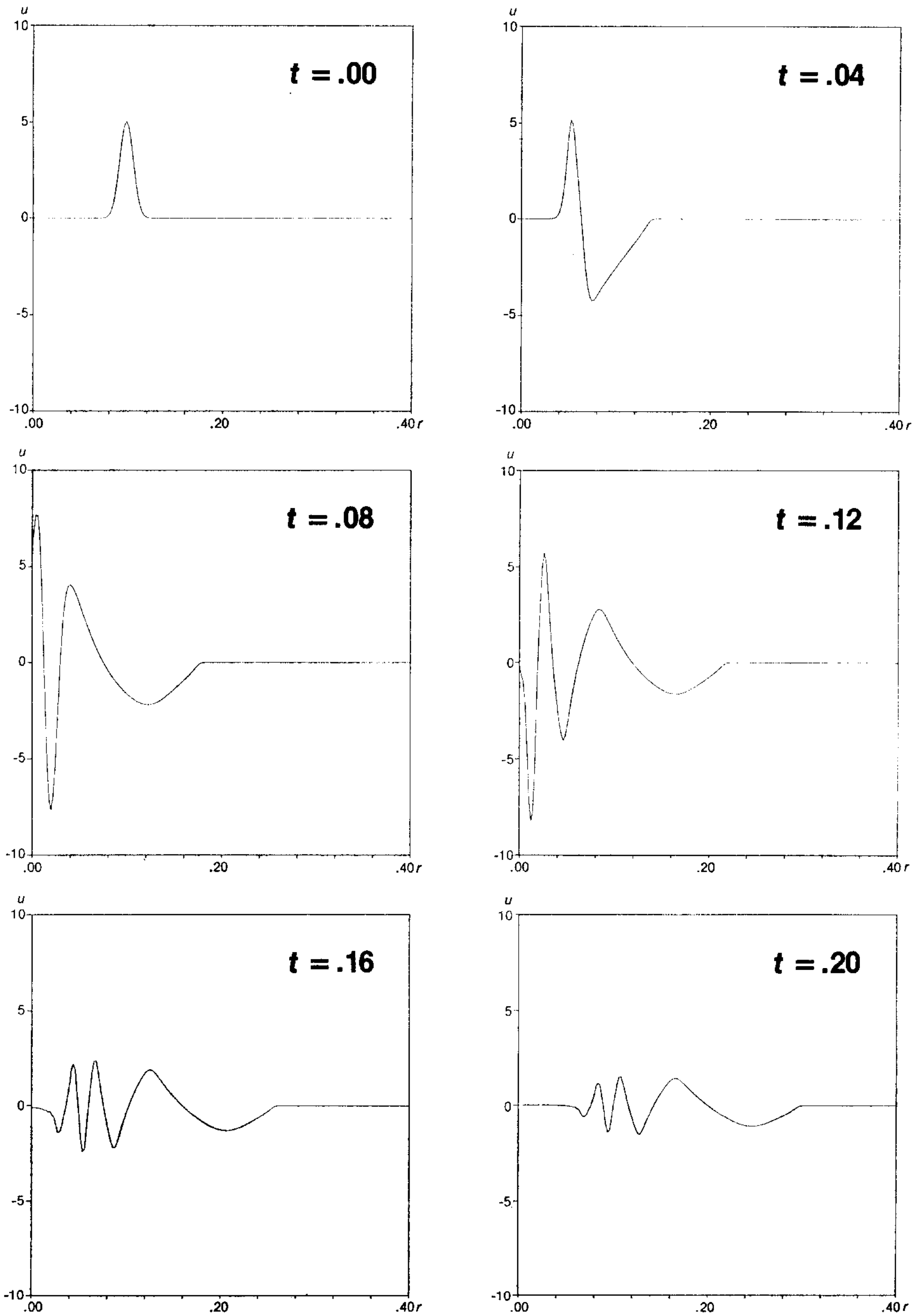

FIG. 1. A typical solution at successive times for $G'(u) = u^7$.

That is, $v_j^0 = f(j\,\Delta r)$ and $v_j^1 = f((j+1)\,\Delta r)$. The energy is $E_n = 67.85$. In Fig. 1 we can see the initial data immediately breaking up into outgoing and incoming parts. The incoming part resolves into oscillations, particularly as it approaches the origin. By the time $t = 0.16$ the initial data have had a chance to reflect at the origin, all the oscillations seem to have become outgoing and no new oscillations appear. It is also clear that the peaks are moving at approximately speed one. Most of the clearly identifiable peaks move at precisely speed one, according to numerical data (not shown); this would be expected in the linear case. These general characteristics of the evolution in time are common to all the solutions which we ran. So in the succeeding figures, we only present solutions at the time $t = 0.20$ after all the oscillations have been produced.

Our computations for this example also show that the fourth term in E_n drops by a factor of 100 from $t = 0.120$ to $t = 0.144$ and thereafter becomes even smaller. This shows that the wave becomes linear as soon as the initial data has reflected at the origin. The computations also show that the first term in E_n, the kinetic energy, approaches exactly $\frac{1}{2}E_n$ ("equipartition of energy"). We have also used the solution at times $t = 0.160$ and 0.162 as initial data to solve the linear equation ($G = 0$) and we find a difference of less than 1 % from the nonlinear case.

Next we study the effect of the nonlinear term on the solution. In Fig. 2 we present the graphs of the solutions at time $t = 0.20$ for the initial data $\phi = 0$ and $\Psi = 100h(r)$ for six different equations: $G'(u) = 0$, u^3, u^5, u^7, u^9 and $\sinh(5u) - 5u$. The very simple linear wave is almost mimicked in the cubic case in spite of amplitudes significantly larger than one. In fact, the maximum amplitude in all space and time, $\max_{x,t} | u(x,t)|$, is 49.98 for $G'(u) = 0$, 47.59 for $G'(u) = u^3$, 19.09 for u^5, 9.74 for u^7, 6.56 for u^9, and 3.72 for $\sinh(5u) - 5u$. All these maxima occur at $r = \Delta r = 0.002$ and near $t = 0.100$. Thus the amplitudes decrease with p and the number of oscillations increase with p. In these respects the graphs suggest that the last example may be regarded as a high power. The energies are $E_n = 31.2934$ for the powers and $E_n = 31.3173$ for the last case.

In the left half of Fig. 3 are the solutions for $t = 0.20$ for the initial data $\phi = h(r)$ and $\Psi = 0$ for the three nonlinear terms $G'(u) = u^p$, $p = 3, 5, 7$. In the right half of Fig. 3 are the solutions for $t = 0.20$ for the initial data

$$\begin{aligned} \phi &= h(r) \quad \text{for} \quad r \geqslant 0.10, \\ &= 5 \quad \text{for} \quad r \leqslant 0.10, \end{aligned} \qquad \Psi = 0. \tag{10}$$

In each case there is a stark contrast between the waves for $p = 3$ and $p = 7$. The cubic case ($p = 3$) is only slightly different from the linear case (graph omitted). For initial data of moderate size the cubic nonlinearity seems to have only a minor effect. The nonlinear effect seems to become pronounced only around $p = 4$ or 5. Of course, for very small initial data the waves will look linear for any value of p.

We offer the following heuristic explanation for these results. A solution with

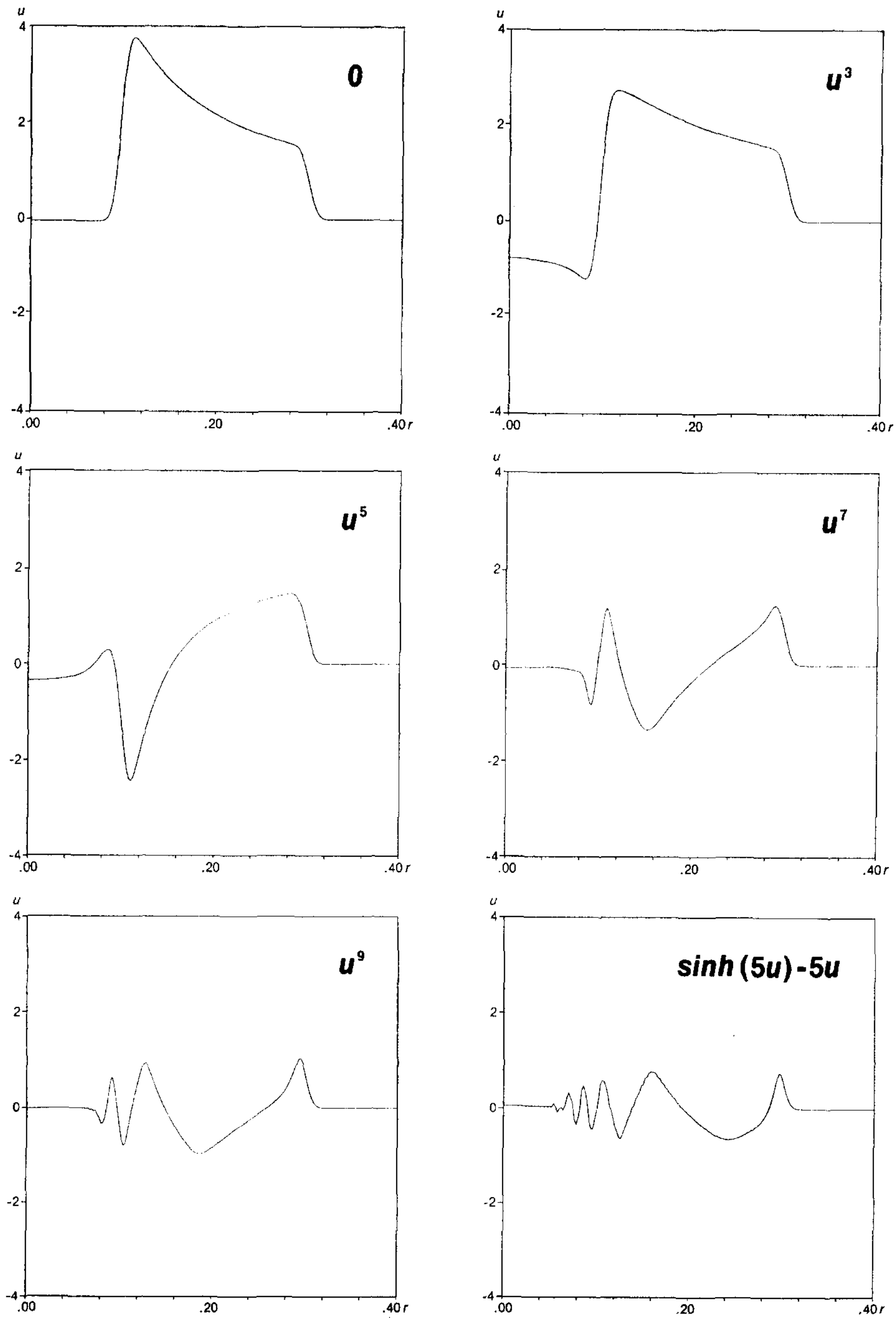

FIG. 2. Solutions for $G'(u) = 0$, u^3, u^5, u^7, u^9 and $\sinh(5u) - 5u$ at time $t = 0.20$ for identical initial data at $t = 0$. The nonlinear term is indicated in the upper-right corner.

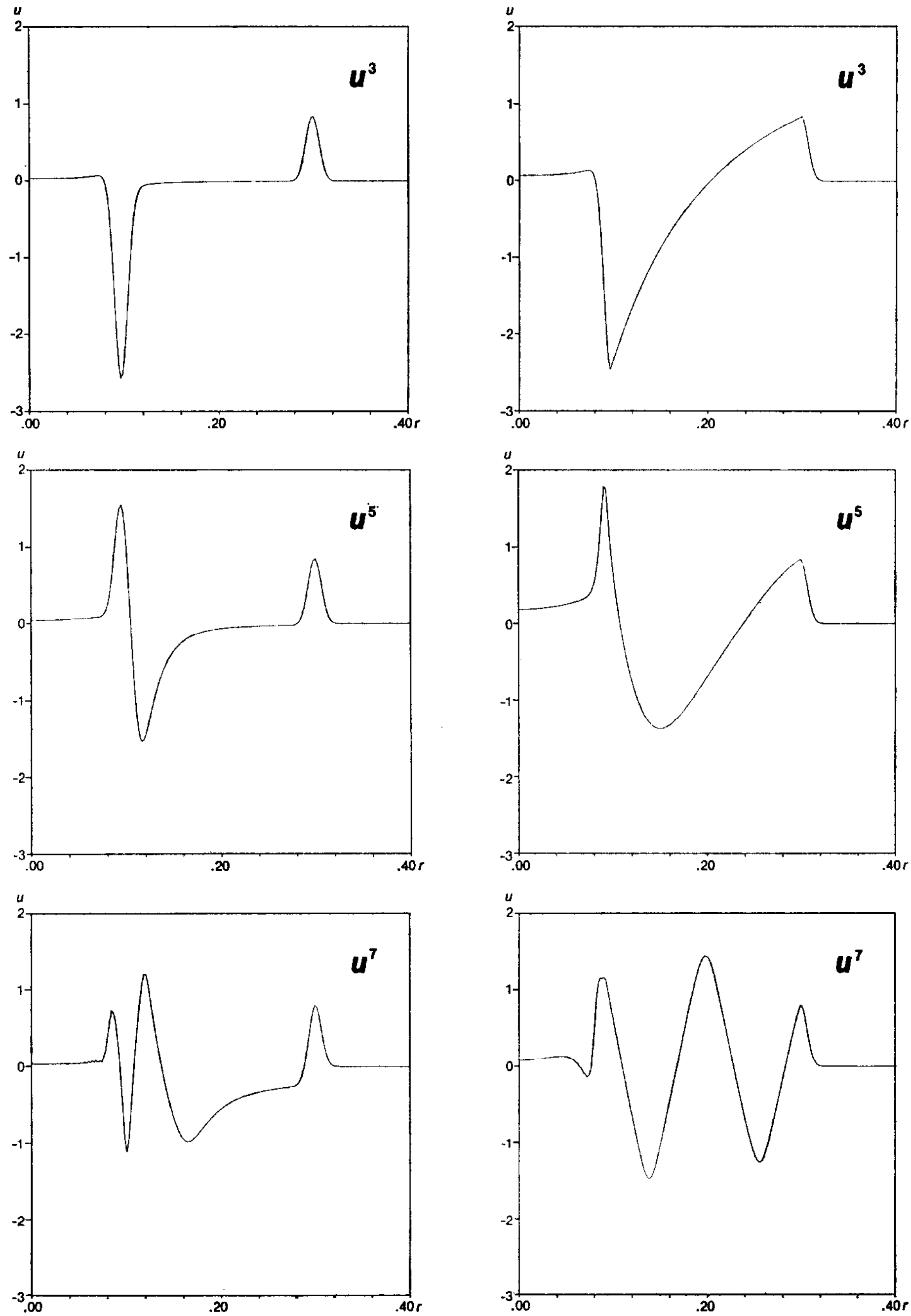

FIG. 3. Solutions for $G'(u) = u^3$, u^5, and u^7 at time $t = 0.20$ for initial data $\phi = h(r)$, $\Psi = 0$ on the left, and for initial data (10) on the right.

energy $E = u_t^2/2 + u^{p+1}/(p+1)$ of the ordinary differential equation $u_{tt} + u^p = 0$ (p odd) is periodic with period

$$T = 4(2E)^{-1/2} ((p+1)E)^{1/(p+1)} \int_0^1 (1 - w^{p+1})^{-1/2} \, dw.$$

Hence, if $E > 1$ is fixed, the period decreases as p increases. The kinetic (u_{tt}) and interaction (u^p) terms provide the dominant effect for a while. After a certain time T, the waves which were initially generated have had a chance to reflect at the origin and become outgoing. As soon as they do so, they spread out spatially. The term Δu becomes important and seems to swamp the interaction term. From that time on, the wave appears linear and so it decays like $O(t^{-3/2})$.

A heuristic explanation for the decay rate is as follows. Each term in the energy (3) is bounded by the constant E. In particular, the third term looks like

$$\int u^2 \, dx \sim \text{const } t^3 M^2(t)$$

for large t since the radius of the support of u increases proportionately to t. Thus $\max_x |u|^2$ looks like $O(t^{-3})$. Arguing similarly with the last term in (3), we can say that (for $G'(u) = u^p$)

$$\int u^{p+1} \, dx \sim \text{const } t^3 M^{p+1}(t)$$

is bounded. But, in a region where $|u| > 1$, $|u|^{p+1}$ increases with p. Hence the amplitude should be smaller for larger p. The above estimates assume that, unlike waterspouts on the ocean surface, the waves have no localized large amplitudes. We do not know how to justify this assumption rigorously unless $p < 5$.

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